Single quantum realization of a collision of two Bose–Einstein condensates

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We propose a method for simulating a single realization of a collision of two Bose–Einstein condensates. Recently in Ziń *et al.* (Phys. Rev. Lett. **94**, 200401 (2005)) we introduced a quantum model of an incoherent elastic scattering in a collision of two counter-propagating atomic Gaussian wavepackets. Here show that this model is capable of generating the data that can be interpreted as results of a single collision event. We find a range of parameters, including relative velocity, population and the size of colliding condensates, where the structure of hallo of scattered atoms in a single realization strongly differs from that averaged over many realizations.

According to the standard interpretation of quantum mechanics the single particle wavefunction $\psi(\mathbf{r},t)$ describes probability amplitude and $|\psi(\mathbf{r},t)|^2$ - probability density for finding this particle around $\mathbf{r}$ at time t. To make a comparison between theory and experiment one needs to repeat the same measurement many times since the agreement can be checked only on the statistical level. Recent developments in the physics of Bose Einstein condensates of neutral atom gasses [1] created an extremely interesting situation where already a single measurement, due to large number of particles involved, reveals some statistical aspects. One prominent example is given by two overlapping independent Bose-Einstein condensates (each of them in the Fock state), where every single measurement of the system reveals an interference pattern. At the same time however, no pattern is present in the average over many measurements [2, 3]. This example shows that even for many particle systems there is a substantial difference between results of a single experiment and the statistical average (over many repetitions of the experiment). The latter, in analogy with the single particle case, is much easier to handle theoretically, the former, more straightforward for experimentalist, is a real challenge for theoretician. Nevertheless, in some cases theoretical solutions are possible. Javanainen and Yoo [3] were able to prove that indeed interference fringes are present in single measurement of overlapping independent condensates. Bach and Rzążewski [4] studied quantum correlations in atomic systems, and noted that single measurements data contain information about these correlations. Dziarmaga [5] proved that any Bogoliubov vacuum can be brought to a diagonal form in a time-dependent orthonormal basis. This diagonal form is tailored for simulations of quantum measurements on an excited condensate. They illustrated their method using example of the phase imprinting of a dark soliton. Here we examine a similar approach for the case when the substantial depletion of the condensate occurs.

Bragg diffraction divides a Bose-Einstein condensate into two initially overlapping components, which immediately start moving away from each other with high relative momentum. Elastic collisions between pairs of atoms from distinct wave packets cause losses that can significantly deplete the condensate. Recently [6, 7] we introduced the quantum model of two counter-propagating atomic Gaussian wave packets to study such collisions and the transition from spontaneous to stimulated regime. Within this model correlation functions of all orders are accessible. It is then not surprising that it allows for a next conceptual step - reconstruction of the result in a single realization of the experiment. The proposed method is valid in the regime of bosonic stimulation and we show the relation to the previous analytical methods [5], but also how it can be used to test the validity of stochastic method [8].

We consider a half-collision of two Bose-Einstein condensates initiated by Bragg scattering. Hamiltonian of the system in second quantization reads, see [6, 7]

$$\begin{aligned}\hat{\mathrm{H}} &= -\int d^3r\, \hat{\Psi}^\dagger(\mathbf{r},t)\frac{\hbar^2\nabla^2}{2m}\hat{\Psi}(\mathbf{r},t) \\ &+ \frac{g}{2}\int \mathrm{d}^3r\, \hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t), \qquad (1)\end{aligned}$$

where $\hat{\Psi}(\mathbf{r},t)$ is the bosonic field operator. Atoms interact via two-body contact interaction, determined by the single coupling constant g. As the system consists of two counter-propagating, highly occupied atomic wave-packets and the "sea" of unoccupied modes, in the spirit of Bogoliubov approximation we decompose the field operator into $c-$number condensate wave-function $\psi_Q(\mathbf{r},t)+\psi_{-Q}(\mathbf{r},t)$ and the quantum field of scattered atoms, $\hat{\delta}(\mathbf{r},t)$, :

$$\hat{\Psi}(\mathbf{r},t) = \psi_Q(\mathbf{r},t)+\psi_{-Q}(\mathbf{r},t)+\hat{\delta}(\mathbf{r},t). \qquad (2)$$

The subscripts $\pm Q$ denote the mean momentum of the colliding wavepackets. In what follows we assume that the condensate wave-function satisfies independently the time-dependent GPE. We insert Eq. (2) into the Hamiltonian (1) and keep only terms, that lead to creation or annihilation of a pair of particles in the $\hat{\delta}$ field of scattered

atoms,

$$\mathrm{H} = -\int \mathrm{d}^3r\, \hat{\delta}^\dagger(\mathbf{r},t)\frac{\hbar^2\nabla^2}{2m}\hat{\delta}(\mathbf{r},t) + g\int \mathrm{d}^3r\, \hat{\delta}^\dagger(\mathbf{r},t)\hat{\delta}^\dagger(\mathbf{r},t)\psi_Q(\mathbf{r},t)\psi_{-Q}(\mathbf{r},t) + \mathrm{H.c.} \tag{3}$$

Further simplification is possible in the regime when the collisional time $t_C = \sigma/(\hbar Q/m)$ is much shorter than the linear disspersion time $t_{LD} = m\sigma^2/\hbar$ and the nonlinear disspersion time $t_{ND} = \sqrt{\pi^{3/2}m\sigma^5/gN}$ [7]. Here σ and $N/2$ is the radius and number of particles in each of the colliding partners. In this regime neither dispersion nor the nonlinearity produce any noticeable effects on the condensate part during the collision and the change of the condensates' shape can be neglected. To model the condensates we use spherically symmetric Gaussian wave-functions

$$\psi_{\pm Q}(\mathbf{r},t) = \sqrt{\frac{N}{2\pi^{3/2}\sigma^3}}\exp\left[\pm iQx_1 - \frac{i\hbar tQ^2}{2m}\right]\times \times\exp\left[-\frac{1}{2\sigma^2}\left(\left(x_1 \mp \frac{\hbar Qt}{m}\right)^2 + x_2^2 + x_3^2\right)\right], \tag{4}$$

where $\mathbf{r} = (x_1, x_2, x_3)$. In this approximation the Heisenberg equation for the dimensionless field operator $\hat{\delta}(\mathbf{r},t)\,\sigma^{3/2}\exp(i\beta t/2) \to \hat{\delta}(\mathbf{r},t)$ in the dimensionless units ($t/t_C \to t$, $x_i/\sigma \to x_i$ (for $i = 1,2,3$), and with parameters $t_{LD}/t_C = \beta$ and $(t_{LD}/t_{ND})^2 = \alpha$) reads

$$i\beta\partial_t\hat{\delta}(\mathbf{r},t) = -\frac{1}{2}\left(\nabla^2 + \beta^2\right)\hat{\delta}(\mathbf{r},t) + \alpha e^{-r^2-t^2}\hat{\delta}^\dagger(\mathbf{r},t). \tag{5}$$

We decompose the field operator into a basis modes of spherically symmetric harmonic oscillator [9],

$$\hat{\delta}(\mathbf{r},t) = \sum_{n,l,m} R_{n,l}(r)Y_{lm}(\theta,\phi)\hat{b}_{n,l,m}(t), \tag{6}$$

as discussed in [6], where we solved the system of linear equations for operators $b_{n,l,m}(t)$.

Here, following the ideas introduced in [10], we take the advantage of the fact that the Hamiltonian of the system is quadratic in creation and anihilation operators and reduce the problem to independently evolving modes that experience squeezing. To do so we map all three quantum numbers n, l, m into single index ($[n, l, m] \to [i]$) and write the solution of the evolution equation (5) in a compact form

$$\vec{b}(t) = C\vec{b}(0) + S\vec{b}^\dagger(0), \tag{7}$$

where vector $\vec{b} = (\hat{b}_1, \hat{b}_2, \ldots)^T$ and C and S are time dependent (evolution) matrices. According to [10], C and S can be written as $C = UC_DV^\dagger$ and $S = US_DV^{\mathrm{T}}$, were U and V are unitary and C_D and S_D are diagonal with non-negative eigenvalues. Hence it is possible to define new initial modes (vectors) $\vec{a}(0) = V^\dagger\vec{b}(0)$ and new final modes: $\vec{a}(t) = U^\dagger\vec{b}(t)$ that are related by simple diagonal transformation

$$\vec{a}(t) = C_D\vec{a}(0) + S_D\vec{a}^\dagger(0) \tag{8}$$

with a condition $C_{D,ii}^2 - S_{D,ii}^2 \equiv c_i^2 - s_i^2 = 1$. The choice of the time t is crucial as it determines both initial and final sets of modes in Eq. (8). It follows from Eq. (8) that the system is in the multimode squeezed state [10], with atomic field operator

$$\hat{\delta}(\mathbf{r},t) = \sum_i \psi_i(\mathbf{r},t)\hat{a}_i(t), \tag{9}$$

where $\hat{a}_i(t) = c_i\hat{a}_i(0) + s_i\hat{a}_i^\dagger(0)$. Notice that the density matrix is these modes diagonal [11]

$$\rho(\mathbf{r}_1,\mathbf{r}_2,t) = \langle\hat{\delta}^\dagger(\mathbf{r}_1,t)\hat{\delta}(\mathbf{r}_2,t)\rangle = \sum_i \psi_i^*(\mathbf{r}_1,t)\psi_i(\mathbf{r}_2,t)s_i^2.$$

To find the set of $\psi_i(\mathbf{r},t)$ we first solve the linear equation (5), obtain the general solution of the form (7) and evaluate the density matrix. Finally, following reference [5], we find the squeezed modes by diagonalization of the density matrix and the anomalous matrix.

For highly populated modes ($s_i \gg 1$) we have $c_i \simeq s_i$. Then the anihilation operators can be rewritten as

$$\hat{a}_i(t) \simeq s_i\left(\hat{a}_i(0) + \hat{a}_i^\dagger(0)\right) = \sqrt{2\langle n_i\rangle}\hat{x}_i(0), \tag{10}$$

where the hermitian operator $\hat{x}_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2}$ is the field quadraure and $\langle n_i\rangle = s_i^2$ is the mode occupation. If we assume that highly populated modes have dominant contribution to the field operator, then

$$\hat{\delta}(\mathbf{r},t) \simeq \sum_{i:\langle n_i\rangle\gg 1}\sqrt{2\langle n_i\rangle}\hat{x}_i(0)\psi_i(\mathbf{r},t). \tag{11}$$

Within this approximation any product of operators $\hat{\delta}$ depends only on $\hat{x}_i$ operators (in general it could also depend on $\hat{p}_i = (\hat{a}_i - \hat{a}_i^\dagger)/i\sqrt{2}$) and the n-th order correlation function,

$$\rho_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t) = \langle\hat{\delta}^\dagger(\mathbf{r}_1,t)\ldots\hat{\delta}^\dagger(\mathbf{r}_n,t)\hat{\delta}(\mathbf{r}_n,t)\ldots\hat{\delta}(\mathbf{r}_1,t)\rangle = \langle\{\hat{\delta}^\dagger(\mathbf{r}_1,t)\ldots\hat{\delta}^\dagger(\mathbf{r}_n,t)\hat{\delta}(\mathbf{r}_n,t)\ldots\hat{\delta}(\mathbf{r}_1,t)\}_{\mathrm{sym}}\rangle, \tag{12}$$

where the symmetrization is performed with respect to $\hat{x}_i$ and $\hat{p}_i$. Thus the evaluation of the quantum average in Eq. (12) is equivalent to the procedure in which we replace the operators $\hat{x}_i$'s and $\hat{p}_i$'s with c-number variables x_i and p_i and perform an integral of the product of $\delta(\mathbf{r},t) = \sum_i\sqrt{2\langle n_i\rangle}x_i\psi_i(\mathbf{r},t)$ function with the Wigner distribution of the vacuum state,

$$\rho_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t) = \int\mathrm{d}\{x_i\}\int\mathrm{d}\{p_i\}W(\{x_i\},\{p_i\}) \delta^*(\mathbf{r}_1,t)\ldots\delta^*(\mathbf{r}_n,t)\delta(\mathbf{r}_n,t)\ldots\delta(\mathbf{r}_1,t) = \int\mathrm{d}\{x_i\}\int\mathrm{d}\{p_i\}W(\{x_i\},\{p_i\})|\delta(\mathbf{r}_1,t)|^2\ldots|\delta(\mathbf{r}_n,t)|^2. \tag{13}$$

Here

$$W(\{x_i\},\{p_i\}) = \prod_i \frac{1}{\pi} \exp\left(-x_i^2 - p_i^2\right). \quad (14)$$

The integrals over p_i's can be calculated explicitly and we obtain

$$\rho_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t) = \int \mathrm{d}\{x_i\} \prod_i \frac{1}{\sqrt{\pi}} \exp\left(-x_i^2\right) \times$$
$$|\delta(\mathbf{r}_1,t)|^2 \cdot \ldots \cdot |\delta(\mathbf{r}_n,t)|^2 \quad (15)$$

We stress that any correlation function can be evaluated according to the procedure described above and conclude that

$$P(\{x_i\}) = \prod_i \frac{1}{\sqrt{\pi}} \exp\left(-x_i^2\right) \quad (16)$$

gives the probability distribution of quadratures' amplitudes x_i's.

Thereby we generated a scheme for calculating the single realization of the scattering experiment: we randomly choose the set of values x_i with probabilities $P(\{x_i\})$ and construct the $\delta(\mathbf{r},t)$ function. To find density distribution in a single realization we take $|\delta(\mathbf{r},t)|^2$. This result is analogous to that obtained in [5]. Here we see the analogy to the so-called Wigner stochastic method [8]. In our method we randomly choose the amplitudes of the quadratures of the macroscopically populated modes and to calculate any moment of the field operator $\hat{\delta}(\mathbf{r},t)$ we simply perform a statistical average over variables x_i. In contrary to the Wigner method, we do not introduce any external noise to the dynamics of the system. The detailed discussion will be given elsewhere.

Now we apply the method derived above to plot the density of the hallo of scattered particles, in a single realization of the collision. Since the density of particles is measured after the free expansion of the atomic cloud, it represents the momentum distribution of atoms. Hence we show the results in the momentum space, $|\delta(\mathbf{k},t)|^2$, where $\mathbf{k}$ is a wave-vector of the scattered atoms. We present both the cut as well as the column density, i.e. $\int dk_z |\delta(\mathbf{k},t)|^2$, see Fig. 1. All the data corresponds to the time t, long after the collision was completed.

As shown in [7] the angular size of the coherence of scattered atoms is proportional to $1/\beta$. We expect that by decreasing β, and thus increasing the range of coherence, at some stage we should observe clear anisotropy (spikes) in the distribution of scattered atoms. The spikes would be an effect of the bosonic stimulation of scattering into highly occupied modes. On the other hand, as β increases, the range of coherence decreases. Then, scattering of separate atoms is practically independent. In such a case we expect that all the momentum modes available to scattered atoms should be almost uniformly occupied. These predictions agree very well with the results obtained in the numerical simulations.

Figure 1 shows the cut for $k_z = 0$ as well as the column density of atoms scattered out from the condensate for three different values of β. The ratio α/β, which is a measure of bosonic enhancement [7], is in all cases the same, and equals $\alpha/\beta = 5$. This value corresponds to the regime of strong bosonic enhancement, where condition of high mode occupation is fulfilled. For the fixed radius of the condensate one can vary the value of β by changing the mutual velocity of the colliding clouds. In Fig. 1, case a), the collision can be regarded as slow and we observe a clear speckle structure. For larger values of β (Figure 1b and 1c) the distribution becomes more and more uniform and resembles an average over many realizations.

Notice that the increase of β can be achieved also by changing the size of the clouds, which is proportional to number of atoms. That explains the experimental results obtained in [12]. They are consistent with our data. While their condensates where rather slow the size (3×10^7 atoms) was very big, giving $\beta \simeq 100$. In this regime we expect that after coarse graining introduced by the CCD camera resolution, the momentum density of scattered atoms would be highly uniform. This is exactly what we see in experimental data shown in Fig.2b of [12]. If one would intend to see the speckles, he should reduce the velocity of colliding condensates and/or reduce their size by decreasing the number of atoms. Alternatively one can use tighter traps. According to our estimates it should be feasible to reduce the value of β to reach the $\beta \simeq 20$ region.

Due to the large number of atoms, even the data obtained in a single experiment has some attributes of the statistical average [4]. It is illustrated in the last column of Fig.1. In this case we fixed the angle between $\mathbf{k}_1 = [\phi', \theta', Q]$ and $\mathbf{k}_2 = [\phi' + \phi, \theta', Q]$, keeping their length equal to Q and calculated the average $\int \delta^*(\mathbf{k}_1)\delta(\mathbf{k}_2)\, d\Omega'$ for three different realizations (dashed lines) and compared it with quantum mechanical average $\langle \delta^\dagger(\mathbf{k}_1)\delta(\mathbf{k}_2)\rangle$ (solid line). The agreement is striking and getting better with increasing value of β (when the distribution becomes more uniform).

In summary we show that single realization of the experiment of two colliding condensates contains an interesting information and may differ substantially from the picture obtained after averaging over many realizations. Especially for slow collisions we predict the lumpy structure in the momentum distribution of scattered atoms, with clear spikes even in the column density. The spikes found in this Letter are analogous to the speckles in the light beam produced by a multimode laser [13]. Even closer optical analogy can be found in recent studies of the parametric down conversion [14].

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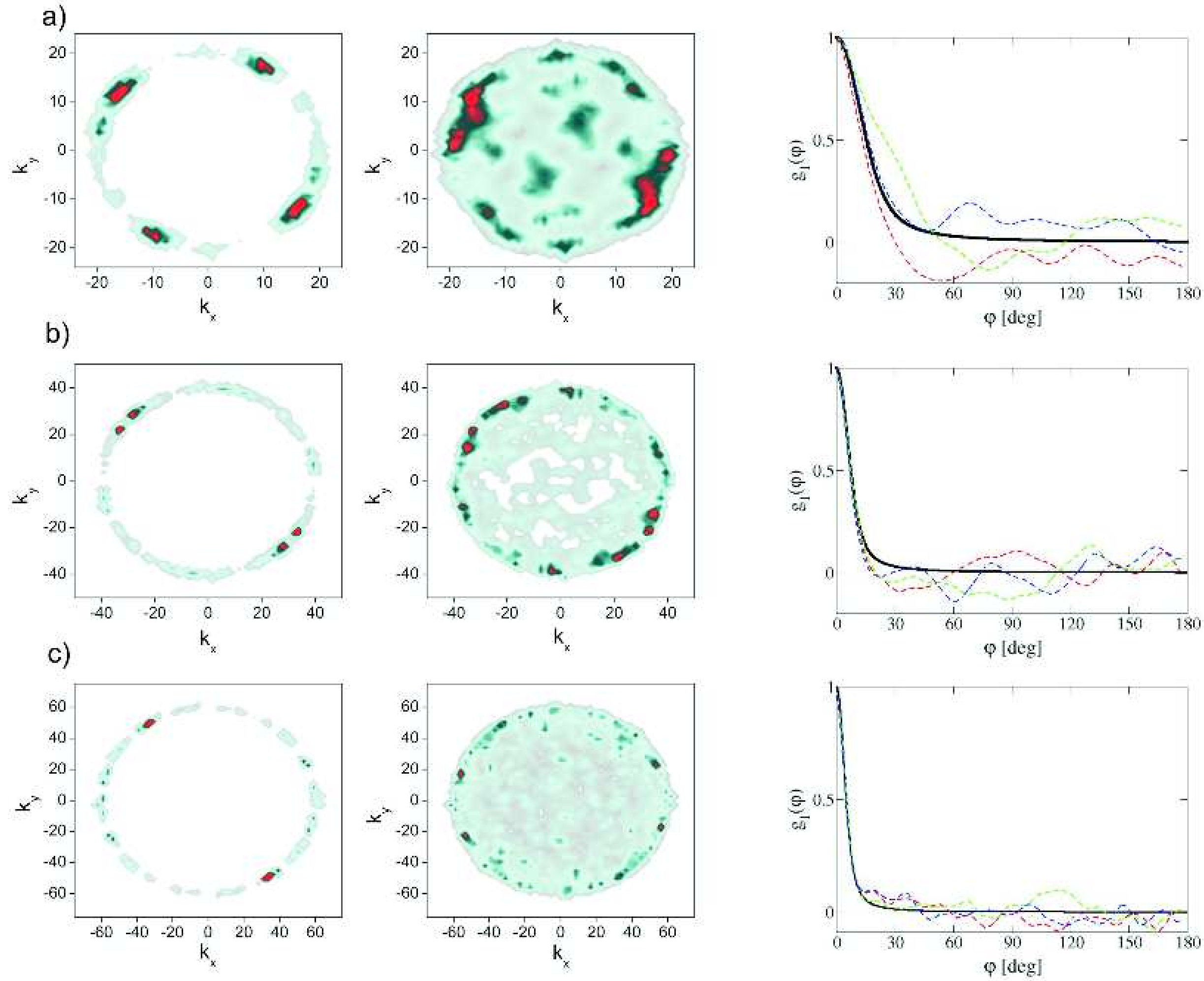

FIG. 1: (Color online): Momentum distribution and normalized density matrix of scattered atoms for three different sets of parameters (β, α); a) (20,100), b) (40,200), c) (60,300). Left panel corresponds to a cut with $k_z = 0$, center to a column density. The right panel shows the comparison of the normalized density matrix averaged over many realizations (solid line) with the three different cases (dashed lines) of the density matrix calculated for a single realization averaged over the full solid angle.

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[2] M. R. Andrews et al., Science 275, 637 (1997), A. Rohrl, M. Naraschewski, A. Schenzle, and H. Wallis, Phys. Rev. Lett. 78, 4143 (1997).

[3] J. Javanainen and S. M. Yoo, Phys. Rev. Lett. 76, 161 (1996).

[4] R. Bach and K. Rzążewski, Phys. Rev. Lett. 92, 200401 (2004).

[5] J. Dziarmaga and K. Sacha, J. Phys. B, **3**, 57 (2006).

[6] P. Ziń *et al.*, Phys. Rev. Lett. **94**, 200401 (2005).

[7] P. Ziń, J. Chwedeńczuk and M. Trippenbach, Phys. Rev. A **73**, 033602 (2006)

[8] A. A. Norrie, R.J. Ballagh and C.W. Gardiner, Phys. Rev. Lett. **94**, 040401 (2005).

[9] M. Abramovich, I. A. Stegun, "Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables", Dover Publications 1974.

[10] Samuel L. Braunstein, Phys. Rev. A, **71**, 055801 (2005).

[11] We work in Heisenberg picture, so all averages are calculated in the vacuum state $|\Omega\rangle$: $a_i|\Omega\rangle = 0$ for all i's.

[12] J. M. Vogels, K. Xu, and W. Ketterle, Phys. Rev. Lett. **89**, 020401 (2002)

[13] C. E. Webb, J. D. C Jones, "Handbook of Laser Technology and Applications", IoP Publishing, Bristol 2004.

[14] E. Brambilla *et al.*, Phys. Rev. A **69**, 023802 (2004)